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Supermodularity in Various Partition Problems ⋆

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Abstract. Supermodular and submodular functions have attracted a great deal of attention since the seminal paper of Lovász. Recently, supermodular functions were studied in the context of some optimal partition problems. We completely answer a question arisen there whether a certain partition function is supermodular.

Key words: Partition, Supermodularity, Sum-partition

1. Introduction

Consider a finite set N of n numbers $\theta^1, \theta^2, \dots, \theta^n$. If $\theta^i \geqslant 0$ for all i or $\theta^i \leqslant 0$ for all i, we call N *one-sided* (or *1-sided*, for short). A partition $\pi = (\pi_1, \pi_2, \dots, \pi_p)$ partitions N into p disjoint parts. In the *unlabeled* partition problem, π is invariant under permutations; in the *labeled* version, π is not. $(|\pi_1|, |\pi_2|, \dots, |\pi_p|)$ is referred to as the *shape* of π. Let Π denote the set of partitions under consideration. If Π is defined by a shape $(n_1, n_2, \dots, n_p)$, $\sum_{i=1}^{p} n_i = n$, then we have a *single-shape-partition* problem. If Π is defined by lower bounds $\{\ell_i\}$ and upper bounds $\{u_i\}$ such that $\ell_i \leqslant n_i \leqslant u_i$ for all $i = 1, 2, \dots, p$ and $\sum_{i=1}^{p} \ell_i \leqslant n \leqslant \sum_{i=1}^{p} u_i$, we have the *bounded-shape-partition* problem. If Π is an arbitrary set of shapes, then we have the *constrained-shape-partition* problem.

From now on we consider Π and $\theta^1, \theta^2, \dots, \theta^n$ as given. In the sum-partition problem, a partition $\pi \in \Pi$ is projected into a point $\theta(\pi) = (\sum_{j\in\pi_1} \theta^j, \sum_{j\in\pi_2} \theta^j, \dots, \sum_{j\in\pi_p} \theta^j)$ in $\Re^p$. Let P^Π, called the *partition polytope*, denote the convex hull of $\theta(\pi)$ for all $\pi \in \Pi$. It is of interest to characterize the vertices of P^Π since if the objective function is quasi-convex, then there exists a vertex representing an optimal partition.

Define $S = \{1, 2, \dots, p\}$. A set function $f(I)$, $I \subseteq S$, is called *supermodular* if for all subsets I and J of S,

$$f(I) + f(J) \leqslant f(I \cap J) + f(I \cup J).$$

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Define set function

$$\theta_*^\Pi(I) = \min_{\pi\in\Pi} \sum_{i\in I} \sum_{j\in\pi_i} \theta^j.$$

It was shown in [1] that whether $\theta_*^\Pi(I)$ is supermodular is important to the study of P^Π. In particular, the following table shows our knowledge on the supermodularity properties of the function $\theta_*^\Pi(I)$ in various partition problems:

labeled	shape	θ	supermodularity
yes	single	general	yes
yes	bounded	1-sided	yes
yes	bounded	general	?
yes	constrained	1-sided	?
yes	constrained	general	?
no	single	1-sided	?
no	single	general	?
no	bounded	1-sided	?
no	bounded	general	?
no	constrained	1-sided	?
no	constrained	general	?

The first case was proved in [1], and then extended to the second case in [2]. In this paper we answer the supermodularity question in every other case.

2. Supermodularity

Assume that

$$\theta^1 \leqslant \theta^2 \leqslant \cdots \leqslant \theta^n. \tag{2.1}$$

Note that (2.1) implies that

$$\sum_{j=u+1}^{u+w} \theta^j \leqslant \sum_{j=v+1}^{v+w} \theta^j \text{ for nonnegative integer } u,\ v,\ \text{and } w \text{ with } u \leqslant v. \tag{2.2}$$

Let $\overline{I}$ denote the complement of set I. For a labeled bounded-shape partition with bounds $\{\ell_i\}$ and $\{u_i\}$, define

$$L(I) = \sum_{i\in I} \ell_i, \quad \text{and} \quad U(I) = \sum_{i\in I} u_i.$$

Define

$$L^*(I) = \max\{L(I),\ n - U(\overline{I})\}, \quad \text{and} \quad U^*(I) = \min\{U(I),\ n - L(\overline{I})\}.$$

Also define

$$n(I) = \max\left\{\sum_{i\in I}|\pi_i| : \pi = (\pi_1, \pi_2, \cdots, \pi_p) \in \Pi,\ \sum_{i\in I}\sum_{j\in\pi_i}\theta^j = \theta_*^\Pi(I)\right\}.$$

Then $L^*(I)$ is a lower bound for the number of elements of N that belong to the parts in I subject to the fact that not too many elements are left for the parts in $\overline{I}$, and $U^*(I)$ is an upper bound for the number of elements of N that belong to the parts in I subject to the fact that enough elements must be left for the parts in $\overline{I}$. Moreover, $n(I)$ is the maximum number of elements of N that belong to the parts in I when a partition achieves the value $\theta_*^\Pi(I)$.

It is obvious that $L(I) + L(J) = L(I \cap J) + L(I \cup J)$ and $U(I) + U(J) = U(I \cap J) + U(I \cup J)$ hold for all I, J. Moreover,

$$L^*(I) \leqslant n(I) \leqslant U^*(I)$$

holds for all I, and if $I \subseteq J$, then

$$L^*(I) \leqslant L^*(J), \quad n(I) \leqslant n(J), \quad \text{and} \quad U^*(I) \leqslant U^*(J).$$

LEMMA 2.1. *For any subsets I, J of S,*

(i) $U^*(I) + U^*(J) \geqslant U^*(I \cap J) + U^*(I \cup J)$, *and*

(ii) $L^*(I) + L^*(J) \leqslant L^*(I \cap J) + L^*(I \cup J)$.

Proof. We only prove (i). The proof of (ii) is similar.

(a) $U^*(I \cup J) = U(I \cup J)$. Then $U^*(I) = U(I)$, $U^*(J) = U(J)$, $U^*(I \cap J) = U(I \cap J)$. It follows $U^*(I) + U^*(J) = U(I) + U(J) = U(I \cap J) + U(I \cup J) = U^*(I \cap J) + U^*(I \cup J)$.

(b) $U^*(I \cup J) = n - L(\overline{I \cup J})$, but $U^*(I) = U(I)$, $U^*(J) = U(J)$. Then $U^*(I \cap J) = U(I \cap J)$. Since $U^*(I \cup J) \leqslant U(I \cup J)$, the proof of (a) still works with the last equality replaced by '$\geqslant$'.

(c) $U^*(I \cup J) = n - L(\overline{I \cup J})$, $U^*(I) = n - L(\overline{I})$, but $U^*(J) = U(J)$. Then $U^*(I \cap J) = U(I \cap J)$. $U^*(I) + U^*(J) - U^*(I \cap J) - U^*(I \cup J) = n - L(\overline{I}) + U(J) - U(I \cap J) - (n - L(\overline{I \cup J})) = U(J \setminus I) - L(J \setminus I) \geqslant 0$. (The case $U^*(I \cup J) = n - L(\overline{I \cup J})$, $U^*(I) = U(I)$, $U^*(J) = n - L(\overline{J})$ is similar.)

(d) $U^*(I \cup J) = n - L(\overline{I \cup J})$, $U^*(I) = n - L(\overline{I})$, $U^*(J) = n - L(\overline{J})$, and $U^*(I \cap J) = U(I \cap J)$. Since $U^*(I \cap J) = U(I \cap J)$, we have $U(I \cap J) \leqslant n - L(\overline{I \cap J})$. Then $U^*(I) + U^*(J) - U^*(I \cap J) - U^*(I \cup J) = n - L(\overline{I}) + n - L(\overline{J}) - U(I \cap J) - (n - L(\overline{I \cup J})) = n - L(\overline{J}) + L(\overline{I \cup J}) - L(\overline{I}) - U(I \cap J) = n - L(\overline{I \cap J}) - U(I \cap J) \geqslant 0$.

(e) $U^*(I \cup J) = n - L(\overline{I \cup J})$, $U^*(I) = n - L(\overline{I})$, $U^*(J) = n - L(\overline{J})$, and $U^*(I \cap J) = n - L(\overline{I \cap J})$. Then $U^*(I) + U^*(J) - U^*(I \cap J) - U^*(I \cup J) = L(\overline{I \cap J}) + L(\overline{I \cup J}) - L(\overline{I}) - L(\overline{J}) = 0$. ∎

THEOREM 2.2. *Let Π be a set of labeled bounded-shape partitions. Then θ_*^Π is supermodular.*

Proof. Let I, J be two subsets of S. Without loss of generality, assume $U^*(J) \leqslant U^*(I)$. Let k be the index such that $\theta^1, \theta^2, \ldots, \theta^k \leqslant 0$ and $\theta^{k+1}, \theta^{k+2}, \ldots, \theta^n > 0$.

(i) $U^*(I \cup J) \leqslant k$. Then $n(I \cap J) = U^*(I \cap J)$, $n(J) = U^*(J)$, $n(I) = U^*(I)$, and $n(I \cup J) = U^*(I \cup J)$. So by Lemma 2.1, we have

$$n(I) + n(J) \geqslant n(I \cap J) + n(I \cup J). \tag{2.3}$$

Hence

$$\begin{aligned}
\theta_*^\Pi(J) - \theta_*^\Pi(I \cap J) &= \sum_{j=n(I\cap J)+1}^{n(J)} \theta^j \\
&\leqslant \sum_{j=n(I\cap J)+1}^{n(I\cap J)+n(I\cup J)-n(I)} \theta^j \text{ (by (2.3) and the fact } \theta^j \text{ involved are } \leqslant 0) \\
&\leqslant \sum_{j=n(I)+1}^{n(I\cup J)} \theta^j \text{(by (2.2) and by the fact that } n(I \cap J) \leqslant n(I)) \\
&= \theta_*^\Pi(I \cup J) - \theta_*^\Pi(I).
\end{aligned}$$

(ii) $U^*(J) \leqslant U^*(I) \leqslant k < U^*(I \cup J)$. Then $n(I \cap J) = U^*(I \cap J)$, $n(J) = U^*(J)$, $n(I) = U^*(I)$, and $n(I \cup J) = \max\{L^*(I \cup J),\ k\} \leqslant U^*(I \cup J)$. Therefore (2.3) is still true and the proof in (i) still works.

(iii) $U^*(J) \leqslant k < U^*(I)$. Then $n(I \cap J) = U^*(I \cap J)$, $n(J) = U^*(J)$, $n(I) = \max\{L^*(I),\ k\}$, and $n(I \cup J) = \max\{L^*(I \cup J),\ k\}$. Since $n(I) \geqslant k$, we have $\theta^j > 0$ whenever $j \geqslant n(I) + 1$. Since $n(I \cup J) \geqslant n(I)$, we have $\sum_{j=n(I)+1}^{n(I\cup J)} \theta^j \geqslant 0$. Hence

$$\begin{aligned}
\theta_*^\Pi(J) - \theta_*^\Pi(I \cap J) = \sum_{j=n(I\cap J)+1}^{n(J)} \theta^j \leqslant 0 \leqslant \sum_{j=n(I)+1}^{n(I\cup J)} \theta^j = \\
= \theta_*^\Pi(I \cup J) - \theta_*^\Pi(I).
\end{aligned}$$

(iv) $U^*(I \cap J) \leqslant k < U^*(J)$. Then $n(I \cap J) = U^*(I \cap J) \leqslant k$, $n(J) = \max\{L^*(J),\ k\}$, $n(I) = \max\{L^*(I),\ k\}$, and $n(I \cup J) = \max\{L^*(I \cup J),\ k\}$. We claim that

$$n(I) + n(J) \leqslant k + n(I \cup J). \tag{2.4}$$

If $n(I \cup J) = k$, then since $L^*(J) \leqslant L^*(I \cup J) \leqslant k$ and since $L^*(I) \leqslant L^*(I \cup J) \leqslant k$, we have $n(I) = n(J) = k$ and $n(I)+n(J) = k+n(I \cup J)$. If $n(I \cup J) = L^*(I \cup J) > k$, then it is easily seen that $n(I)+n(J) \leqslant k+n(I \cup J)$ except when $n(I) = L^*(I)$, $n(J) = L^*(J)$; but then $n(I) + n(J) = L^*(I) + L^*(J) \leqslant L^*(I \cap J) + L^*(I \cup J) \leqslant k + n(I \cup J)$. Hence

$$\begin{aligned}
\theta_*^{\Pi}(J) - \theta_*^{\Pi}(I \cap J) &= \sum_{j=n(I \cap J)+1}^{n(J)} \theta^j \\
&\leqslant \sum_{j=k+1}^{n(J)} \theta^j (\text{ since the } \theta^j \text{ involved are } > 0) \\
&\leqslant \sum_{j=k+1}^{k+n(I \cup J)-n(I)} \theta^j \text{ (by (2.4))} \\
&\leqslant \sum_{j=n(I)+1}^{n(I \cup J)} \theta^j \text{ (by (2.2) and by the fact of } k \leqslant n(I)) \\
&= \theta_*^{\Pi}(I \cup J) - \theta_*^{\Pi}(I).
\end{aligned}$$

(v) $U^*(I \cap J) > k$. Then $n(I \cap J) = \max\{L^*(I \cap J),\ k\}$, $n(J) = \max\{L^*(J),\ k\}$, $n(I) = \max\{L^*(I),\ k\}$, and $n(I \cup J) = \max\{L^*(I \cup J),\ k\}$. We claim that

$$n(I) + n(J) \leqslant n(I \cap J) + n(I \cup J). \tag{2.5}$$

The proof is similar to that of (2.4) and is omitted here. Hence

$$\begin{aligned}
\theta_*^{\Pi}(J) - \theta_*^{\Pi}(I \cap J) &= \sum_{j=n(I \cap J)+1}^{n(J)} \theta^j \\
&\leqslant \sum_{j=n(I \cap J)+1}^{n(I \cap J)+n(I \cup J)-n(I)} \theta^j (\text{ by (2.5) and by the fact that the } \theta^j \\
&\qquad \text{involved are } > 0) \\
&\leqslant \sum_{j=n(I)+1}^{n(I \cup J)} \theta^j \text{ (by (2.2) and by the fact that } n(I \cap J) \leqslant n(I)) \\
&= \theta_*^{\Pi}(I \cup J) - \theta_*^{\Pi}(I).
\end{aligned}$$

■

THEOREM 2.3. *Let* Π *be an unlabeled single-shape partition defined by the shape* $(n_1, n_2, \ldots, n_p)$ *and suppose* θ *is 1-sided. Then* θ_*^{Π} *is supermodular.*

Proof. Without loss of generality, order the p sizes in the given shape into

$$n_1 \geqslant n_2 \geqslant \cdots \geqslant n_p \text{ if } \theta^i \leqslant 0 \quad \text{for all } i,$$

and

$$n_1 \leqslant n_2 \leqslant \cdots \leqslant n_p \text{ if } \theta^i \geqslant 0 \quad \text{for all } i.$$

Since Π is unlabeled, we can consider any mapping of the p sizes to the p parts. Let $n_I = \sum_{i=1}^{|I|} n_i$ for all I. Consider I and J. Then

$$\begin{aligned}
\theta_*^{\Pi}(I) &= \sum_{i=1}^{n_I} \theta^i, \\
\theta_*^{\Pi}(J) &= \sum_{i=1}^{n_J} \theta^i, \\
\theta_*^{\Pi}(I \cap J) &= \sum_{i=1}^{n_{I\cap J}} \theta^i, \text{ and} \\
\theta_*^{\Pi}(I \cup J) &= \sum_{i=1}^{n_{I\cup J}} \theta^i.
\end{aligned}$$

Clearly,

$$n_I + n_J = n_{I\cap J} + n_{I\cup J} \tag{2.6}$$

holds for all I, J. By (2.6), by (2.2), and by the fact that $n_{I\cap J} \leqslant n_J$, we have $\theta_*^{\Pi}(I) - \theta_*^{\Pi}(I \cap J) = \sum_{n_{I\cap J}+1}^{n_I} \theta^i \leqslant \sum_{i=n_J+1}^{n_{I\cup J}} \theta^i = \theta_*^{\Pi}(I \cup J) - \theta_*^{\Pi}(J)$. ∎

Next we show that for labeled constrained partition with 1-sided θ, θ_*^{Π} is not supermodular. Let $p = 4$, $n = 8$, $\Pi = \{(3, 1, 3, 1),\ (1, 4, 1, 2)\}$, $\theta^1 = \theta^2 = \cdots = \theta^8 = 1$, $I = \{1, 2\}$, $J = \{2, 3\}$. Then

$$\begin{aligned}
&\theta_*^{\Pi}(I) = 3 + 1 = 4, \qquad \theta_*^{\Pi}(J) = 1 + 3 = 4,\\
&\theta_*^{\Pi}(I \cap J) = 1, \quad \text{and} \quad \theta_*^{\Pi}(I \cup J) = 1 + 4 + 1 = 6.
\end{aligned}$$

Since the sum of the first two is greater than the sum of the last two, θ_*^{Π} is not supermodular.

For unlabeled constrained partition with 1-sided θ, consider $p = 4$, $n = 16$, $\Pi = \{(1, 5, 5, 5),\ (3, 3, 4, 6)$ and their permutations$\}$, $\theta^1 = \theta^2 = \cdots = \theta^{16} = 1$, $I = \{1, 2\}$, $J = \{1, 3\}$. Then

$$\begin{aligned}
&\theta_*^{\Pi}(I) = 1 + 5(\text{or } 3 + 3) = 6, \qquad \theta_*^{\Pi}(J) = 1 + 5(\text{or } 3 + 3) = 6,\\
&\theta_*^{\Pi}(I \cap J) = 1, \quad \text{and} \quad \theta_*^{\Pi}(I \cup J) = 3 + 3 + 4 = 10.
\end{aligned}$$

Again the sum of the first two is greater than the sum of the last two, hence θ_*^Π is not supermodular.

Note that the negative results for the two 1-sided cases of course extend to general θ. We next show that for unlabeled single-shape partition with general θ, θ_*^Π is not supermodular.

Let $p = 4, n = 6, \Pi = \{(1, 1, 1, 3)$ and its permutations$\}, \theta^1 = \theta^2 = \theta^3 = -1$, $\theta^4 = \theta^5 = \theta^6 = 1$, $I = \{1, 2\}$, $J = \{1, 3\}$. Then

$$\theta_*^\Pi(I) = (-1) + (-1)(\text{ or } (-1) + (-1) + (-1) + 1) = -2,$$
$$\theta_*^\Pi(J) = (-1) + (-1)(\text{ or } (-1) + (-1) + (-1) + 1) = -2,$$
$$\theta_*^\Pi(I \cap J) = (-1) + (-1) + (-1) = -3, \text{ and}$$
$$\theta_*^\Pi(I \cup J) = (-1) + (-1) + (-1) = -3.$$

Again the sum of the first two is greater than the sum of the last two, hence θ_*^Π is not supermodular. This negative result extends to unlabeled bounded-shape partition with general θ.

Finally, we show that for unlabeled bounded-shape partition with 1-sided θ, θ_*^Π is not supermodular. Let $p = 4$, $n = 10$, $\ell_1 = 1$, $u_1 = 4$, $\ell_2 = \ell_3 = \ell_4 = 2$, $u_2 = u_3 = u_4 = 3$, $\theta^1 = \theta^2 = \cdots = \theta^{10} = 1$, $I = \{1, 2\}$, and $J = \{1, 3\}$. Since the partition is unlabeled, we can consider any mapping between the four bound-intervals and the four parts. Thus

$$\theta_*^\Pi(I) = \theta_*^\Pi(J) = 4$$

by assigning the interval $[1, 4]$ to π_1 and the interval $[2, 3]$ to π_2, π_3 and π_4. For $\theta_*^\Pi(I)$, we choose $n_1 = n_2 = 2$ and $n_3 = n_4 = 3$ (the choice for $\theta_*^\Pi(J)$ is analogous).

$$\theta_*^\Pi(I \cap J) = 1$$

by assigning the interval $[1, 4]$ to π_1 and the interval $[2, 3]$ to π_2, π_3 and π_4. For $\theta_*^\Pi(I \cap J)$, we choose $n_1 = 1$ and $n_2 = n_3 = n_4 = 3$. Furthermore,

$$\theta_*^\Pi(I \cup J) = 6$$

by assigning the interval $[1, 4]$ to π_4 and the interval $[2, 3]$ to π_1,π_2 and π_3. For $\theta_*^\Pi(I \cup J)$, we choose $n_1 = n_2 = n_3 = 2$ and $n_4 = 4$. Since

$$\theta_*^\Pi(I) + \theta_*^\Pi(J) = 8 > \theta_*^\Pi(I \cap J) + \theta_*^\Pi(I \cup J) = 7,$$

θ_*^Π is not supermodular.

3. Conclusion

We have the following new table:

labeled	shape	θ	supermodularity
yes	single	general	yes
yes	bounded	1-sided	yes
yes	bounded	general	yes
yes	constrained	1-sided	no
yes	constrained	general	no
no	single	1-sided	yes
no	single	general	no
no	bounded	1-sided	no
no	bounded	general	no
no	constrained	1-sided	no
no	constrained	general	no

All ‘constrained’ cases answer no, and yes in the ‘unlabeled’ case implies the same for the corresponding ‘labeled’ case. There is no other obvious pattern. Most of the ‘1-sided’ cases answer yes, but there is exception. Most of the ‘single’ cases answer yes, but there is exception. Most of the ‘bounded’ cases answer the same as their corresponding ‘single’ cases, but there is exception.

References

1. Gao, B., Hwang, F.K., Li, W.W.-C. and Rothblum, U.G. (1999), Partition polytopes over 1-dimensional points, *Math. Prog.* 85: 335–362.
2. Hwang, F.K. and Rothblum, U.G., Partitions: Clustering and Optimality, (in progress).
3. Lovász, L., *Submodular functions and complexity*, in A. Bachem et al. (eds.), Mathematical Programming: The State of the Art, pp. 235–257.